Grupa A, Pismeni ispit iz Matematike II, 19.09.2013. ispit pisati isključivo hemijskom olovkom

1. Izračunati dužinu luka polukubičnog paraboloida $y^2 = (x-1)^3$ između tački $A(2;-1)$ i $B(5;-8)$.

2. Izračunati $\iint\limits_D y\,\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) : x^2+y^2 \le 1,\ x^2+y^2 \le 2x,\ y \ge 0\}$.

3. Metodom diferenciranja po parametru izračunati integral $\int\limits_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}}\mathrm{d}x \quad (a^2<1)$ (mala pomoć: možda ćete naći korisno da u rješavanju integrala iskoristie smjene $x = \sin t$ ili $\operatorname{tg} t = z$).

4. Izračunati krivoliniski integral druge vrste $I = \oint\limits_C (y-z)\mathrm{d}x + (z-x)\mathrm{d}y + (x-y)\mathrm{d}z$ gdje je C krug $x^2+y^2+z^2 = a^2$ $(a>0)$, $y = x\operatorname{tg}\alpha$, $(0<\alpha<\frac{\pi}{2})$ uzet u smjeru suprotnom kretanju kazaljnke na satu ako se posmatra sa pozitivnog dijela x-ose.

Grupa B, Pismeni ispit iz Matematike II, 19.09.2013. ispit pisati isključivo hemijskom olovkom

1. Izračunati dužinu luka jednog svoda cikloide $x = a(t-\sin t)$, $y = a(1-\cos t)$ (za jedan svod cikloide parametar t uzima vrijednosti od 0 do 2π).

2. Izračunati $I = \iint\limits_G \left(x + \frac{y^2}{x^2}\right)\mathrm{d}x\mathrm{d}y$ gdje je $G = \{(x,y) : x^2+y^2-2ax \le 0, a>0\}$.

3. Metodom diferenciranja po parametru izračunati integral $\int\limits_0^1 \frac{\operatorname{arc\,tg} ax}{x\sqrt{1-x^2}}\mathrm{d}x$ (mala pomoć: možda ćete naći korisno da u rješavanju integrala iskoristie smjene $x = \sin t$ ili $\operatorname{tg} t = z$).

4. Odrediti površinu koju cilindar $x^2+y^2 = ax$ isjeca na lopti $x^2+y^2+z^2 = a^2$ iznad ravni Oxy.

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Izračunati dužinu luka polukubičnog paraboloida $y^2=(x-1)^3$ između tački $A(2;-1)$ i $B(5;-8)$.

Rj.

<u>Dužinu luka krive $y=f(x)$ između tački $(a;f(a))$ i $(b;f(b))$ računamo pomoću formule</u>

$$\ell=\int_a^b\sqrt{1+[f'(x)]^2}\,dx.$$

U našem slučaju

$$y^2=(x-1)^3$$

$$y=\pm\sqrt{(x-1)^3}=\pm(x-1)^{\frac{3}{2}} \Rightarrow y'=\pm\frac{3}{2}(x-1)^{\frac{1}{2}}$$

$$L_{AB}=\int_{x_A}^{x_B}\sqrt{1+(y')^2}\,dx=\int_2^5\sqrt{1+\frac{9}{4}(x-1)}\,dx=\frac{1}{2}\int_2^5\sqrt{9x-5}\,dx=$$

$$=\left|\begin{array}{l} d(9x-5)=9\,dx \\ dx=\frac{1}{9}d(9x-5)\end{array}\right|=\frac{1}{18}\int_2^5(9x-5)^{\frac{1}{2}}d(9x-5)=$$

$$=\frac{1}{27}(9x-5)^{\frac{3}{2}}\Big|_2^5=\ldots=\frac{80}{27}\sqrt{10}-\frac{13}{27}\sqrt{13}$$

traženo rješenje

$$\approx 7{,}6337$$

Izračunati dužinu luka jednog svoda cikloide $x=a(t-\sin t)$, $y=a(1-\cos t)$ (za jedan svod cikloide parametar t uzima vrijednosti od 0 do 2π).

Rj.

Dužina luka krive $\begin{cases} x=x(t) \\ y=y(t) \\ t_1 \le t \le t_2 \end{cases}$ se računa po formuli

$$\ell = \int_{t_1}^{t_2} \sqrt{(\dot{x})^2+(\dot{y})^2}\, dt .$$

U našem slučaju

$$\dot{x} = \frac{dx}{dt} = a(1-\cos t) \; ; \quad \dot{y} = \frac{dy}{dt} = a\sin t$$

$$\sqrt{\dot{x}^2+\dot{y}^2}\, dt = \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\, dt =$$

$$= a\sqrt{2(1-\cos t)}\, dt = a\sqrt{4\sin^2\frac{t}{2}}\, dt = 2a\sin\frac{t}{2}\, dt$$

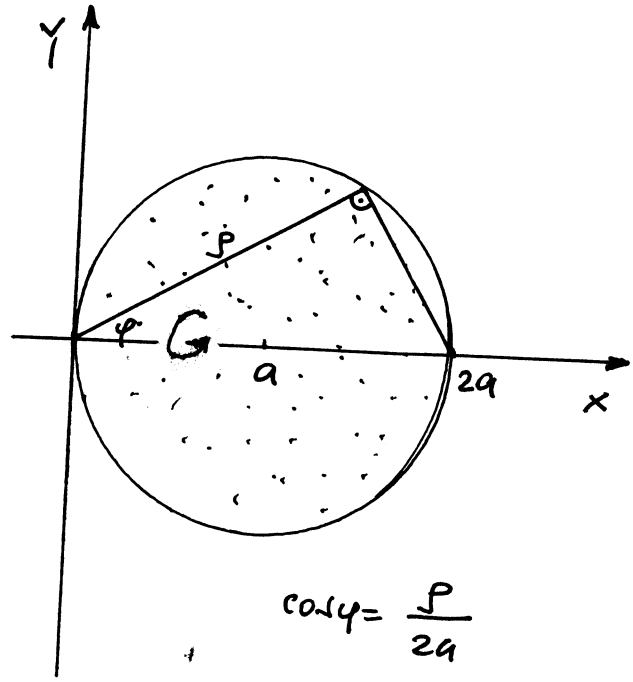
$$L = 2a\int_0^{2\pi} \sin\frac{t}{2}\, dt = 4a\int_0^{2\pi} \sin\frac{t}{2}\, d\frac{t}{2} = -4a\cos\frac{t}{2}\Big|_0^{2\pi} = \ldots = 8a$$

traženo rješenje

(#) Izračunati $I=\iint\limits_G \left(x+\frac{y^2}{x^2}\right)dx\,dy$ gdje je

$$G=\{(x,y): x^2+y^2-2ax\le 0,\ a>0\}.$$

Rj. Skicirajmo oblast G



$$x^2+y^2-2ax=0$$

$$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$$

$$(x-a)^2+y^2=a^2$$

krug sa centrom u tački $C(a,0)$ poluprečnika $r=a$

$$\cos\varphi=\frac{\rho}{2a}$$

$$\rho=2a\cos\varphi$$

Uvedimo polarne koordinate

$$x=\rho\cos\varphi$$

$$y=\rho\sin\varphi$$

$$dx\,dy=\rho\,d\rho\,d\varphi$$

$$G \xrightarrow{\text{transformiše}} G': \begin{cases} -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2} \\ 0\le\rho\le 2a\cos\varphi \end{cases}$$

$$I=\iint\limits_G\left(x+\frac{y^2}{x^2}\right)dx\,dy=\left|\begin{array}{l}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{array}\right|=\iint\limits_{G'}\left(\rho\cos\varphi+\frac{\sin^2\varphi}{\cos^2\varphi}\right)\rho\,d\rho\,d\varphi$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\,d\varphi\int_0^{2a\cos\varphi}\rho^2\,d\rho+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\cos^2\varphi}\,d\varphi\int_0^{2a\cos\varphi}\rho\,d\rho=\ldots=\frac{8a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\varphi\,d\varphi+$$

$$+2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\cos^2\varphi}\cdot\cos^2\varphi\,d\varphi=\ldots=a^3\pi+a^2\pi=a^2(a+1)\pi$$ traženo rješenje

(#) Izračunati $\iint\limits_D y\,dx\,dy$ gdje je $D=\{(x,y): x^2+y^2\le 1,\ x^2+y^2\le 2x,\ y\ge 0\}$

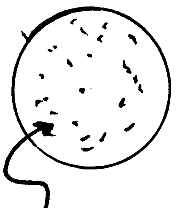
Rj.

$$D=\{(x,y): x^2+y^2\le 1,\quad x^2+y^2\le 2x,\quad y\ge 0\}$$

$x^2+y^2\le 1 \Downarrow$

$1=x^2+y^2$

krug sa centrom u $C(0;0)$ polupr. $r=1$



$x^2+y^2\le 1$

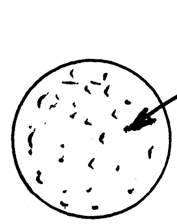
$x^2+y^2\le 2x \Downarrow$

$x^2+y^2=2x$

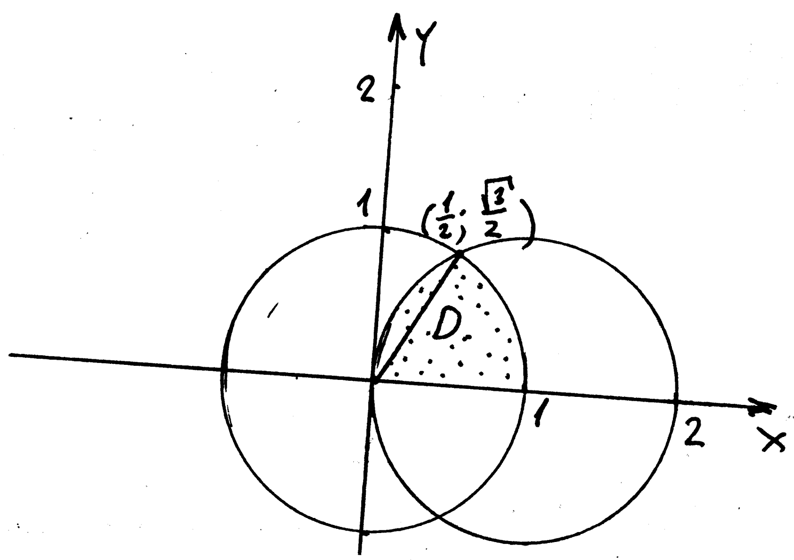
$x^2-2\cdot x+1+y^2=1$

$(x-1)^2+y^2=1$ krug sa centrom u $C(1;0)$ polupr. $r=1$

$x^2+y^2\le 2x$



Skicirajmo oblast D



Ako uvedemo polarne koordinate

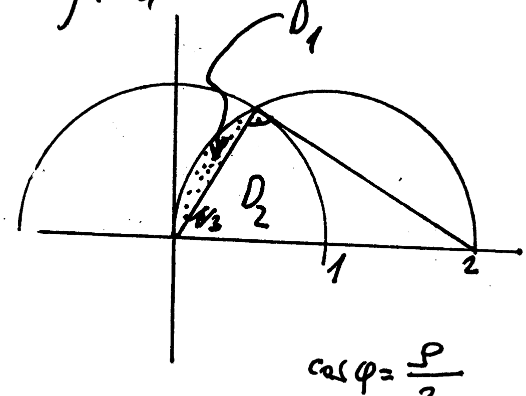
$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$D \xrightarrow{\text{transf.}} D'$

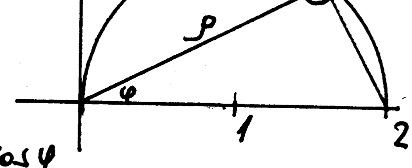
oblast D' možemo podjeliti na dva dijela



$D_2:\begin{cases} 0\le\rho\le 1\\ 0\le\varphi\le\pi/3\end{cases}$

$D_1:\begin{cases} \frac{\pi}{3}\le\varphi\le\frac{\pi}{2}\\ 0\le\rho\le 2\cos\varphi\end{cases}$

$\cos\varphi=\frac{\rho}{2}$



$\rho=2\cos\varphi$

$$\iint_D y\,dx\,dy = \left| \begin{matrix} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{matrix} \right| = \iint_{D'} \rho \sin\varphi \, \rho \, d\rho \, d\varphi =$$

$$= \iint_{D_1 \cup D_2} \rho^2 \sin\varphi \, d\rho \, d\varphi = \iint_{D_1} \rho^2 \sin\varphi \, d\rho \, d\varphi + \iint_{D_2} \rho^2 \sin\varphi \, d\rho \, d\varphi$$

$$\iint_{D_1} \rho^2 \sin\varphi \, d\rho \, d\varphi = \int_{\pi/3}^{\pi/2} \sin\varphi \, d\varphi \int_0^{2\cos\varphi} \rho^2 \, d\rho = \frac{1}{3} \int_{\pi/3}^{\pi/2} \sin\varphi \, \rho^3 \Big|_0^{2\cos\varphi} d\varphi$$

$$= \frac{8}{3} \int_{\pi/3}^{\pi/2} \cos^3\varphi \sin\varphi \, d\varphi = -\frac{8}{3} \int_{\pi/3}^{\pi/2} \cos^3\varphi \, d(\cos\varphi) = -\frac{8}{3} \cdot \frac{1}{4} \cos^4\varphi \Big|_{\pi/3}^{\pi/2} = -\frac{2}{3}\left(0 - \left(\frac{1}{2}\right)^4\right)$$

$$= \frac{2}{3} \cdot \frac{1}{16} = \frac{1}{24}$$

$$\iint_{D_2} \rho^2 \sin\varphi \, d\rho \, d\varphi = \int_0^{\pi/3} \sin\varphi \int_0^1 \rho^2 \, d\rho = \frac{1}{3} \rho^3 \Big|_0^1 \cdot \underbrace{(-\cos\varphi) \Big|_0^{\frac{\pi}{3}}}_{-\left(\frac{1}{2} - 1\right)} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

$$\iint_D y \, dx \, dy = \frac{1}{24} + \frac{1 \cdot 4}{6 \cdot 4} = \frac{5}{24} \text{ traženo rješenje}$$

(#) Metodom diferenciranja po parametru izračunati integral
$$\int_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}}\,dx \quad (a^2<1).$$

Rj: Prisjetimo se

Ako je dat integral $F(\alpha)=\int_a^b f(x,\alpha)\,dx$ u kome granice a i b ne zavise od parametra α i u kome f ima neprekidan parcijalni izvod po α tada
$$F'(\alpha)=\int_a^b f'_\alpha(x,\alpha)\,dx.$$

U našem slučaju f-ja f je $f=\frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}}$

$$\frac{\partial f}{\partial a}=\frac{1}{x^2\sqrt{1-x^2}}\cdot\frac{-2ax^2}{1-a^2x^2}=\frac{-2a}{(1-a^2x^2)\sqrt{1-x^2}}$$

$$F'(a)=\int_0^1\frac{-2a}{(1-a^2x^2)\sqrt{1-x^2}}\,dx=\left|\begin{array}{l} x=\sin t \\ dx=\cos t\,dt \quad x\Big|_0^1 \Rightarrow t\Big|_0^{\pi/2} \\ 1-x^2=1-\sin^2 t=\cos^2 t\end{array}\right|=$$

$$=(-2a)\int_0^{\pi/2}\frac{\cancel{\cos t}\,dt}{(1-a^2\sin^2 t)\cancel{\cos t}}=\left|\begin{array}{ll} \operatorname{tg} t=z & \sin^2 t=\frac{\sin^2 t\ /:\cos^2 t}{\sin^2 t+\cos^2 t\ /:\cos^2 t} \\ t=\operatorname{arctg} z & \sin^2 t=\frac{z^2}{1+z^2} \\ dt=\frac{dz}{1+z^2} & t\Big|_0^{\pi/2}\Rightarrow z\Big|_0^{\infty}\end{array}\right|$$

$$=(-2a)\int_0^\infty\frac{\frac{dz}{1+z^2}}{1-\frac{a^2z^2}{1+z^2}}=(-2a)\int_0^\infty\frac{\frac{dz}{1+z^2}}{\frac{1+z^2-a^2z^2}{1+z^2}}=(-2a)\int_0^\infty\frac{dz}{(1-a^2)z^2+1}=$$

$$= \frac{-2a}{1-a^2} \int_0^{\infty} \frac{dz}{z^2 + \frac{1}{1-a^2}} = \frac{-2a}{1-a^2} \cdot \frac{1}{\sqrt{\frac{1}{1-a^2}}} \operatorname{arc\,tg} \frac{z}{\frac{1}{\sqrt{1-a^2}}} \Bigg|_0^{\infty}$$

$$= \frac{-2a\sqrt{1-a^2}}{1-a^2} \cdot \frac{\pi}{2} = \frac{-a\pi}{\sqrt{1-a^2}}$$

tj. $F_a' = -\pi \frac{a}{\sqrt{1-a^2}}$

$$F(a) = \int F_a' \, da = -\pi \int \frac{a}{\sqrt{1-a^2}} da = \left| \begin{matrix} d(1-a^2) = -2a\,da \\ -a\,da = \frac{1}{2} d(a^2-1) \end{matrix} \right|$$

$$= \frac{\pi}{2} \int (1-a^2)^{-\frac{1}{2}} d(1-a^2) = \frac{\pi}{2} \cdot \frac{(1-a^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = \pi\sqrt{1-a^2} + c$$

$$\left. \begin{aligned} F(a) &= \pi\sqrt{1-a^2} + c \\ F(a) &= \int_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}} dx \end{aligned} \right\} \Rightarrow \left. \begin{aligned} F(0) &= \pi\sqrt{1-0} + c \\ F(0) &= 0 \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow \quad \pi + c = 0 \quad \Rightarrow \quad c = -\pi$$

Prema tome

$$\int_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}} dx = \pi\left(\sqrt{1-a^2} - 1\right)$$

traženo rješenje

(#) Metodom diferenciranja po parametru izračunati integral $\displaystyle\int_0^1 \frac{\operatorname{arctg} ax}{x\sqrt{1-x^2}}\,dx$.

Rj: Prisjetimo se:

Ako je dat integral $F(\alpha)=\int_a^b f(x,\alpha)\,dx$ u kome granice a i b ne zavise od parametra α tada $F'(\alpha)=\int_a^b f'_\alpha(x,\alpha)\,dx$.

U našem slučaju f-ja f je $f(x,a)=\dfrac{\operatorname{arctg} ax}{x\sqrt{1-x^2}}$

$$f'_a=\frac{1}{\cancel{x}\sqrt{1-x^2}}\cdot\frac{\cancel{x}}{1+(ax)^2}=\frac{1}{(1+a^2x^2)\sqrt{1-x^2}}$$

$$F(a)=\int_0^1\frac{\operatorname{arctg} ax}{x\sqrt{1-x^2}}\,dx$$

$$F'_a=\int_0^1\frac{dx}{(1+a^2x^2)\sqrt{1-x^2}}=\left|\begin{array}{ll} x=\sin t & x\Big|_0^1 \Rightarrow t\Big|_0^{\frac{\pi}{2}} \\ dx=\cos t\,dt & \\ \sqrt{1-x^2}=\sqrt{\cos^2 t}=|\cos t| & \end{array}\right|$$

$$=\int_0^{\pi/2}\frac{\cancel{\cos t}\,dt}{(1+a^2\sin^2 t)\cancel{\cos t}}=\int_0^{\pi/2}\frac{dt}{1+a^2\sin^2 t}=\left|\begin{array}{ll} \operatorname{tg} t=z & \sin^2 t=\frac{z^2}{1+z^2} \\ t=\operatorname{arctg} z & \\ dt=\frac{dz}{1+z^2} & t\Big|_0^{\pi/2}\Rightarrow z\Big|_0^{\infty} \end{array}\right|$$

$$=\int_0^\infty\frac{\frac{dz}{1+z^2}}{1+a^2\frac{z^2}{1+z^2}}=\int_0^\infty\frac{\frac{dz}{1+z^2}}{\frac{1+z^2+a^2z^2}{1+z^2}}=\int_0^\infty\frac{dz}{(a^2+1)z^2+1}=$$

$$= \frac{1}{a^2+1}\int_0^\infty \frac{dz}{z^2+\frac{1}{a^2+1}} = \frac{1}{a^2+1}\cdot\frac{1}{\sqrt{\frac{1}{a^2+1}}}\,\mathrm{arc\,tg}\frac{z}{\sqrt{\frac{1}{a^2+1}}}\Bigg|_0^\infty$$

$$= \frac{\sqrt{a^2+1}}{a^2+1}\cdot\frac{\pi}{2} = \frac{\frac{\pi}{2}}{\sqrt{a^2+1}} \implies F'_a = \frac{\pi/2}{\sqrt{a^2+1}}$$

$$F(a) = \frac{\pi}{2}\int\frac{da}{\sqrt{a^2+1}} = \frac{\pi}{2}\ln\left|a+\sqrt{a^2+1}\right| + C$$

Prema tome

$$\left.\begin{aligned} F(a) &= \frac{\pi}{2}\ln\left|a+\sqrt{a^2+1}\right| + C \\ F(a) &= \int_0^1 \frac{\mathrm{arc\,tg}\,ax}{x\sqrt{1-x^2}}\,dx \end{aligned}\right\} \implies \begin{aligned} F(0) &= C \\ F(0) &= 0 \end{aligned} \qquad C=0$$

Prema tome

$$\int_0^1 \frac{\mathrm{arc\,tg}\,ax}{x\sqrt{1-x^2}} = \frac{\pi}{2}\ln\left|a+\sqrt{a^2+1}\right|$$

(#) Izračunati krivolinijski integral druge vrste

$$I = \oint_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$$ gdje je C krug

$x^2+y^2+z^2=a^2$ $(a>0)$, $y = x\,tg\alpha$, $(0<\alpha<\frac{\pi}{2})$ uzet u smjeru suprotnom kretanju kazaljke na satu ako se posmatra sa pozitivnog dijela x-ose.

Rj.

$$C: \begin{cases} x^2+y^2+z^2=a^2, (a>0) \\ y = x\,tg\alpha \quad (0<\alpha<\frac{\pi}{2}) \end{cases}$$

Parametrizujmo krivu C. Kako je $tg\alpha = \frac{\sin\alpha}{\cos\alpha}$ to možemo npr. uzeti $x = a\cos\alpha\sin\varphi$. Tada,

$$y = x\,tg\alpha = a\cancel{\cos\alpha}\sin\varphi\frac{\sin\alpha}{\cancel{\cos\alpha}} = a\sin\alpha\sin\varphi$$

Dalje iz $x^2+y^2+z^2=a^2$ imamo

$$(a\cos\alpha\sin\varphi)^2 + (a\sin\alpha\sin\varphi)^2 + z^2 = a^2$$

$$a^2\underbrace{(\cos^2\alpha+\sin^2\alpha)}_{=1}\sin^2\varphi + z^2 = a^2$$

$$z^2 = a^2 - a^2\sin^2\varphi$$

$$z^2 = a^2(1-\sin^2\varphi) \Rightarrow z = a\cos\varphi$$

Dati krug C ima sljedeću parametrizaciju

$$C: \begin{cases} x = a\cos\alpha\sin\varphi \\ y = a\sin\alpha\sin\varphi \\ z = a\cos\varphi \\ 0 \le \varphi \le 2\pi \end{cases} \qquad \begin{aligned} dx &= a\cos\alpha\cos\varphi\,d\varphi \\ dy &= a\sin\alpha\cos\varphi\,d\varphi \\ dz &= -a\sin\varphi \end{aligned}$$

$$\oint_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz =$$

$$= \int_0^{2\pi} \Big[(a\sin\alpha\sin\varphi - a\cos\varphi)\, a\cos\alpha\cos\varphi + (a\cos\varphi - a\cos\alpha\sin\varphi)\, a\sin\alpha\cos\varphi$$

$$+ (a\cos\alpha\sin\varphi - a\sin\alpha\sin\varphi)(-a)\sin\varphi\Big]\, d\varphi$$

$$= a^2\sin\alpha\cos\alpha\int_0^{2\pi}\sin\varphi\cos\varphi\,d\varphi - a^2\cos\alpha\int_0^{2\pi}\cos^2\varphi\,d\varphi + a^2\sin\alpha\int_0^{2\pi}\cos^2\varphi\,d\varphi$$

$$- a^2\sin\alpha\cos\alpha\int_0^{2\pi}\sin\varphi\cos\varphi\,d\varphi - a^2\cos\alpha\int_0^{2\pi}\sin^2\varphi\,d\varphi + a^2\sin\alpha\int_0^{2\pi}\sin^2\varphi\,d\varphi$$

$$= -a^2\cos\alpha\int_0^{2\pi}\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1}\,d\varphi + a^2\sin\alpha\int_0^{2\pi}\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{=1}\,d\varphi$$

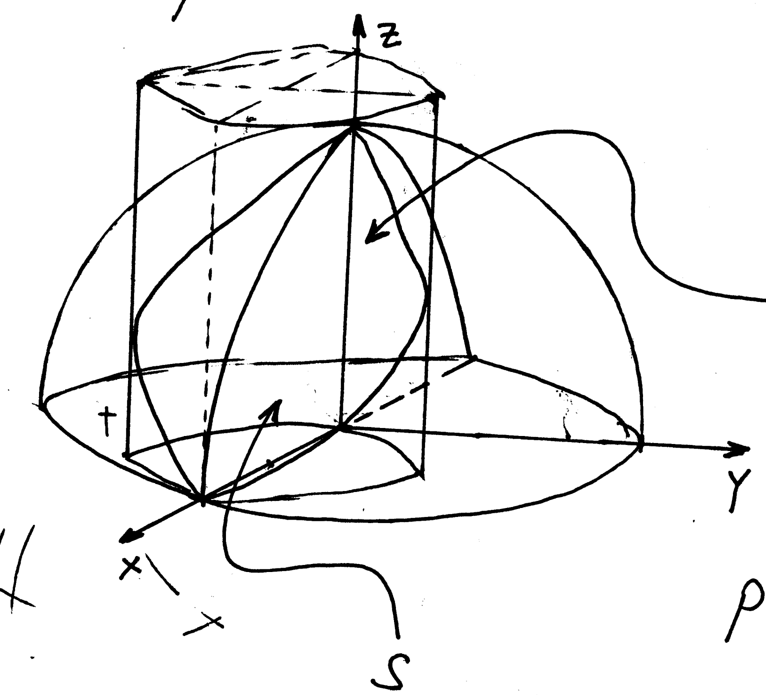
$$= 2\pi a^2(\sin\alpha-\cos\alpha) = 2a^2(\sin\alpha-\cos\alpha)\,\pi$$

traženo rješenje

Odrediti površinu koju cilindar $x^2+y^2=ax$ isjeca na lopti $x^2+y^2+z^2=a^2$ iznad ravni Oxy.

Rj.

Skiciramo sliku



$$x^2+y^2=ax$$

$$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$

Trebamo izračunati površinu dijela lopte koji se nalazi unutar cilindra.

$$P=\iint_S dS=\iint_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$z^2=a^2-x^2-y^2$$

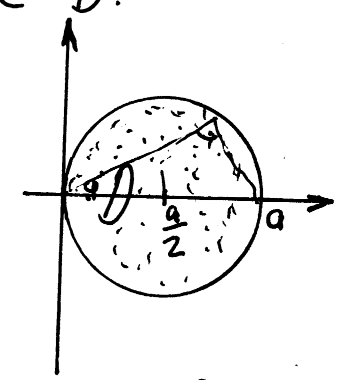
$$z=\pm\sqrt{a^2-x^2-y^2}$$

kako je u pitanju gornji dio polusfere to imamo

$$z=+\sqrt{a^2-x^2-y^2}$$

$$z'_x=\frac{-x}{\sqrt{a^2-x^2-y^2}},\quad z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}},\quad 1+z'^2_x+z'^2_y=\frac{a^2-x^2-y^2+x^2+y^2}{a^2-x^2-y^2}$$

$$P=\iint_D\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dx\,dy=a\iint_D\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$ gdje je D:



Uvedimo polarne koordinate

$$x=\rho\cos\varphi$$
$$y=\rho\sin\varphi$$
$$dx\,dy=\rho\,d\rho\,d\varphi$$

$$D\xrightarrow{\text{transformiše}}D':\begin{cases}0\le\rho\le a\cos\varphi\\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{cases}$$

$$\cos\varphi=\frac{\rho}{a}$$

$$x^2+y^2=\dots=\rho^2$$

$$a\iint_D \frac{dx\,dy}{\sqrt{a^2-(x^2+y^2)}} = \left| \begin{matrix} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{matrix} \right| = a\iint_{D'} \frac{\rho\, d\rho\, d\varphi}{\sqrt{a^2-\rho^2}}$$

$$= a\int_{-\pi/2}^{\pi/2} d\varphi \int_0^{a\cos\varphi} \frac{\rho\, d\rho}{\sqrt{a^2-\rho^2}} = \left| \begin{matrix} d(a^2-\rho^2) = -2\rho\, d\rho \\ \rho\, d\rho = -\frac{1}{2} d(a^2-\rho^2) \end{matrix} \right| =$$

$$= a\int_{-\pi/2}^{\pi/2} d\varphi \int_0^{a\cos\varphi} -\frac{1}{2}(a^2-\rho^2)^{-\frac{1}{2}} d(a^2-\rho^2) = -\frac{1}{2}\cdot 2a \int_{-\pi/2}^{\pi/2} \underbrace{(a^2-\rho^2)^{\frac{1}{2}}\Big|_0^{a\cos\varphi}}_{\underbrace{(a^2-a^2\cos^2\varphi)^{\frac{1}{2}}}_{(a^2\underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi})^{1/2}} - (a^2-0)^{\frac{1}{2}}} d\varphi$$

$$= -a\int_{-\pi/2}^{\pi/2} (a\sin\varphi - a)\, d\varphi$$

$$= -a^2 \int_{-\pi/2}^{\pi/2} (\sin\varphi - 1)\, d\varphi = -a^2 \cdot \Big(\underbrace{-\cos\varphi \Big|_{-\pi/2}^{\pi/2}}_{0} - \underbrace{\varphi \Big|_{-\pi/2}^{\pi/2}}_{\frac{\pi}{2}+\frac{\pi}{2}} \Big) = a^2\pi \quad \text{traženo rješenje}$$